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Three positive solutions of a nonlinear three-point boundary value problem

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Abstract

In this paper, existence criteria for three positive solutions of the nonlinear three-point boundary value problem

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, & t \in (0,1), \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

are established by using the Leggett–Williams fixed point theorem. An example is also included to illustrate the importance of the result obtained.

Keywords: Multi-point boundary value problem; Positive solution; Fixed point

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1. Introduction

Recently, multi-point boundary value problems have received many attentions from many authors [4–6,8–11]. In particular, Ma and Wang [10] obtained the existence of one positive solution for more general three-point boundary value problem

$$u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \alpha u(\eta), \quad 0 < \eta < 1, \tag{1.2}$$

under the assumption that f is either superlinear or sublinear, where (H1) $f \in C([0,\infty), [0,\infty))$; (H2) $h \in C([0,1],[0,\infty))$ and there exists $x_0 \in [0,1]$ such that $h(x_0) > 0$; (H3) $a \in C[0,1]$, $b \in C([0,1],(-\infty,0))$; (H4) $0 < \alpha\phi_1(\eta) < 1$, where ϕ_1 is the unique solution of the linear boundary value problem

$$\begin{cases} \phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, & t \in (0,1), \\ \phi_1(0) = 0, \quad \phi_1(1) = 1. \end{cases} \tag{1.3}$$

Their main tool is a fixed point theorem for mapping defined on Banach spaces with cones [1,3]. For background information and applications of such a fixed point theorem, one can refer to [2,8,12].

In this paper we will consider the existence of at least three positive solutions for the above boundary value problems. Our tool is the well-known Leggett–Williams fixed point theorem [7].

In order to obtain our main results, we need the following concepts and Leggett–Williams fixed point theorem.

Let E be a real Banach space with cone P. A map $\alpha : P \to [0,+\infty)$ is said to be a nonnegative continuous concave functional on P if α is continuous and

$$\alpha\big(tx + (1-t)y\big) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Let a_0, b_0 be two numbers such that $0 < a_0 < b_0$ and α a nonnegative continuous concave functional on P. We define the following convex sets

$$P_{a_0} = \big\{x \in P\colon\ \|x\| < a_0\big\},$$
$$P(\alpha, a_0, b_0) = \big\{x \in P\colon\ a_0 \leqslant \alpha(x),\ \|x\| \leqslant b_0\big\}.$$

Theorem (Leggett–Williams fixed point theorem). *Let $A : \bar{P}_c \to \bar{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leqslant \|x\|$ for all $x \in \bar{P}_c$. Suppose there exist $0 < d_0 < a_0 < b_0 \leqslant c$ such that*

(i) $\{x \in P(\alpha, a_0, b_0)\colon\ \alpha(x) > a_0\} \neq \phi$ *and* $\alpha(Ax) > a_0$ *for* $x \in P(\alpha, a_0, b_0)$;
(ii) $\|Ax\| < d_0$ *for* $\|x\| \leqslant d_0$;
(iii) $\alpha(Ax) > a_0$ *for* $x \in P(\alpha, a_0, c)$ *with* $\|Ax\| > b_0$.

Then A has at least three fixed points x_1, x_2, x_3 in $\bar{P}_c$ satisfying

$$\|x_1\| < d_0, \qquad a_0 < \alpha(x_2), \qquad \|x_3\| > d_0 \quad \textit{and} \quad \alpha(x_3) < a_0.$$

2. Main results

To state the main result of this paper, we need the following lemma, which was established by Ma and Wang [10].

Lemma 2.1. *Assume that* (H3) *holds. Let ϕ_1 and ϕ_2 be the solutions of Eq.* (1.3) *and*

$$\begin{cases} \phi_2''(t)+a(t)\phi_2'(t)+b(t)\phi_2(t)=0, & t\in(0,1),\\ \phi_2(0)=1, \quad \phi_2(1)=0. \end{cases} \tag{2.1}$$

Then

(i) ϕ_1 *is strictly increasing on* $[0,1]$;
(ii) ϕ_2 *is strictly decreasing on* $[0,1]$.

In view of (H2), $h\in C([0,1],[0,\infty))$ and there exists $x_0\in[0,1]$ such that $h(x_0)>0$, and hence we may assume that $x_0\in(0,1)$. Take $\delta\in(0,1/2)$ such that $x_0\in(\delta,1-\delta)$.

For convenience, we let

$$G(t,s)=\frac{1}{\phi_1'(0)}\begin{cases} \phi_1(t)\phi_2(s), & s\geqslant t,\\ \phi_1(s)\phi_2(t), & s\leqslant t, \end{cases}$$

$$D=\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)h(s)\,ds+\frac{\alpha}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)\,ds,$$

and

$$C=\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} G(t,s)p(s)h(s)\,ds+\frac{\alpha\phi_1(\delta)}{1-\alpha\phi_1(\eta)}\int_\delta^{1-\delta} G(\eta,s)p(s)h(s)\,ds,$$

where $p(t)=\exp(\int_0^t a(s)\,ds)$.

For the function $G(t,s)$, it follows from Lemma 2.1 that

$$0\leqslant G(t,s)\leqslant G(s,s), \quad (t,s)\in[0,1]\times[0,1], \tag{2.2}$$

and

$$G(t,s)\geqslant \gamma G(s,s), \quad (t,s)\in[\delta,1-\delta]\times[0,1], \tag{2.3}$$

where $\gamma=\min\{\phi_1(\delta),\phi_2(1-\delta)\}$.

Our main result is the following theorem.

Theorem 2.1. *Assume that* (H1)–(H4) *hold and that there exist numbers d_1 and d_0 with $0<d_0<d_1$ such that*

$$f(u)<\frac{d_0}{D}, \quad u\in[0,d_0], \tag{2.4}$$

and

$$f(u)>d_1/C, \quad u\in\left[d_1,\frac{d_1}{\gamma}\right]. \tag{2.5}$$

Then the boundary value problem (1.1)–(1.2) *has at least three positive solutions if the following condition holds*:

(H5) *There exists a number* $c > d_1/\gamma$ *such that* $f(u) < c/D$ *for* $u \in [0, c]$.

Proof. Let E be the set $C[0,1]$ of all real continuous functions defined on $[0,1]$ endowed with the usual linear structure and the maximum norm. Set

$$P = \{u \in E\colon\ u(t) \geqslant 0,\ t \in [0,1]\}.$$

Then it is easily seen that P is a cone in E. For $u \in P$, define

$$\alpha(u) = \min_{t \in [\delta, 1-\delta]} u(t)$$

and

$$(Au)(t) = \int_0^1 G(t,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds, \quad t \in [0,1].$$

It is easy to check that α is a nonnegative continuous concave functional on P with $\alpha(x) \leqslant \|x\|$ for $x \in P$ and that [10] $A : P \to P$ is completely continuous and fixed points of A are solutions of the boundary value problem (1.1)–(1.2).

We first assert that if there exists a positive number r such that $f(u) < r/D$ for $u \in [0, r]$, then $A : \bar{P}_r \to P_r$.

Indeed, if $u \in \bar{P}_r$, then, in view of Lemma 2.1, for $t \in [0,1]$,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&\quad + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&< \frac{r}{D}\left[\int_0^1 G(t,s)p(s)h(s)\,ds + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\,ds\right] \\
&\leqslant \frac{r}{D}\left[\max_{t\in[0,1]} \int_0^1 G(t,s)p(s)h(s)\,ds + \frac{\alpha}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\,ds\right] \\
&= r.
\end{aligned}$$

Thus, $\|Au\| < r$, i.e., $Au \in P_r$.

Hence, we have shown that if (2.4) and (H5) hold, then A maps $\bar{P}_{d_0}$ into P_{d_0} and $\bar{P}_c$ into P_c.

Next, we assert that $\{u \in P(\alpha, d_1, d_1/\gamma)\colon \alpha(u) > d_1\} \neq \phi$ and $\alpha(Au) > d_1$ for all $u \in P(\alpha, d_1, d_1/\gamma)$.

In fact, the constant function

$$\frac{d_1 + d_1/\gamma}{2} \in \bigl\{u \in P(\alpha, d_1, d_1/\gamma)\colon\ \alpha(u) > d_1\bigr\}.$$

Moreover, for $u \in P(\alpha, d_1, d_1/\gamma)$, we have

$$d_1/\gamma \geqslant \|u\| \geqslant u(t) \geqslant \min_{t\in[\delta,1-\delta]} u(t) = \alpha(u) \geqslant d_1$$

for all $t \in [\delta, 1-\delta]$. Thus, in view of (2.5), we see that

$$\begin{aligned}
\alpha(Au) &= \min_{t\in[\delta,1-\delta]} \Biggl[\int_0^1 G(t,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&\qquad + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds\Biggr] \\
&\geqslant \min_{t\in[\delta,1-\delta]} \int_0^1 G(t,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&\qquad + \frac{\alpha\phi_1(\delta)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&\geqslant \min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} G(t,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&\qquad + \frac{\alpha\phi_1(\delta)}{1-\alpha\phi_1(\eta)} \int_\delta^{1-\delta} G(\eta,s)p(s)h(s)f\bigl(u(s)\bigr)\,ds \\
&> \frac{d_1}{C}\Biggl[\min_{t\in[\delta,1-\delta]} \int_\delta^{1-\delta} G(t,s)p(s)h(s)\,ds \\
&\qquad + \frac{\alpha\phi_1(\delta)}{1-\alpha\phi_1(\eta)} \int_\delta^{1-\delta} G(\eta,s)p(s)h(s)\,ds\Biggr] = d_1,
\end{aligned}$$

as required.

Finally, we assert that if $u \in P(\alpha, d_1, c)$ and $\|Au\| > d_1/\gamma$, then $\alpha(Au) > d_1$.

To see this, suppose $u \in P(\alpha, d_1, c)$ and $\|Au\| > d_1/\gamma$, then, in view of (2.2) and (2.3), we have

$$\begin{aligned}
\alpha(Au) &= \min_{t\in[\delta,1-\delta]}\Bigg[\int_0^1 G(t,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\quad + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f\big(u(s)\big)\,ds\Bigg] \\
&\geqslant \min_{t\in[\delta,1-\delta]}\int_0^1 G(t,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\quad + \frac{\alpha\phi_1(\delta)}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\geqslant \gamma\Bigg[\int_0^1 G(s,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\quad + \frac{\alpha}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f\big(u(s)\big)\,ds\Bigg] \\
&\geqslant \gamma\Bigg[\int_0^1 G(t,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\quad + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f\big(u(s)\big)\,ds\Bigg]
\end{aligned}$$

for $t \in [0,1]$. Thus

$$\begin{aligned}
\alpha(Au) &\geqslant \gamma \max_{t\in[0,1]}\Bigg[\int_0^1 G(t,s)p(s)h(s)f\big(u(s)\big)\,ds \\
&\quad + \frac{\alpha}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f\big(u(s)\big)\,ds\,\phi_1(t)\Bigg] \\
&= \gamma\|Au\| > \gamma\frac{d_1}{\gamma} = d_1.
\end{aligned}$$

To sum up, all the hypotheses of the Leggett–Williams theorem are satisfied by taking $b_0 = d_1/\gamma$. Hence A has at least three fixed points, i.e., the boundary value problem (1.1)–(1.2) has at least three positive solutions u, v and w such that

$$\|u\| < d_0, \qquad d_1 < \min_{t\in[\delta,1-\delta]} v(t), \qquad \|w\| > d_0,$$

and

$$\min_{t\in[\delta,1-\delta]} w(t) < d_1.$$

The proof is complete. □

We remark that the condition (H5) in Theorem 2.1 can be replaced by the following condition (H5′):

$$\lim_{u\to\infty} \frac{f(u)}{u} < \frac{1}{D},$$

which is a special case of (H5).

Corollary 2.1. *If the condition* (H5) *in Theorem* 2.1 *is replaced by* (H5′), *then the conclusion of Theorem* 2.1 *also holds.*

Proof. By Theorem 2.1, we only need to prove that (H5′) implies that (H5) holds, that is, if (H5′) holds, then there exists a number $c > d_1/\gamma$ such that $f(u) < c/D$ for $u \in [0, c]$.

Suppose on the contrary that for any $c > d_1/\gamma$, there exists $u_c \in [0, c]$ such that $f(u_c) \geqslant c/D$. Hence if we choose $c_n > d_1/\gamma$ $(n = 1, 2, \ldots)$ with $c_n \to \infty$, then there exist $u_n \in [0, c_n]$ such that

$$f(u_n) \geqslant \frac{c_n}{D}, \tag{2.6}$$

and so

$$\lim_{n\to\infty} f(u_n) = \infty. \tag{2.7}$$

Since the condition (H5′) holds, then there exists $\tau > 0$ such that

$$f(u) < \frac{u}{D}, \quad u > \tau. \tag{2.8}$$

Hence we have $u_n \leqslant \tau$. Otherwise, if $u_n > \tau$, then it follows from (2.8) that

$$f(u_n) < \frac{u_n}{D} \leqslant \frac{c_n}{D}, \tag{2.9}$$

which contradicts (2.6).

Let

$$M = \max_{u\in[0,\tau]} f(u).$$

Then $f(u_n) \leqslant M$ $(n = 1, 2, \ldots)$, which also contradicts $\lim_{n\to\infty} f(u_n) = \infty$. The proof is complete. □

Example 2.1. Consider the boundary value problem

$$u''(t) - u(t) + f(u(t)) = 0, \quad t \in (0, 1), \tag{2.10}$$

$$u(0) = 0, \qquad u(1) = e^{1/2}u(1/2), \tag{2.11}$$

where

$$f(x)=\begin{cases} x^2, & x\in\left[0, \frac{4(e^2-1)(e^{1/2}+1)}{(2e^{5/4}-2e-e^{1/2}+1)(e^{5/4}-e^{3/4})}\right],\\ lx+k, & x\in\left[\frac{4(e^2-1)(e^{1/2}+1)}{(2e^{5/4}-2e-e^{1/2}+1)(e^{5/4}-e^{3/4})}, +\infty\right), \end{cases}$$

$$l=\frac{e+1}{2(e+1-2e^{1/2})(1+e^{3/2}+e^{1/2})},$$

and

$$k=\left(\frac{4(e^2-1)(e^{1/2}+1)}{(2e^{5/4}-2e-e^{1/2}+1)(e^{5/4}-e^{3/4})}\right)^2 - \frac{4(e+1)(e^2-1)(e^{1/2}+1)}{2(e+1-2e^{1/2})(1+e^{3/2}+e^{1/2})(2e^{5/4}-2e-e^{1/2}+1)(e^{5/4}-e^{3/4})}.$$

A simple calculation shows that

$$\phi_1(t)=\frac{e^{1+t}-e^{1-t}}{e^2-1},\qquad \phi_2(t)=\frac{e^{2-t}-e^t}{e^2-1},\qquad \phi_1'(0)=\frac{2e}{e^2-1},\quad \text{and}$$
$$p(t)\equiv 1.$$

Take $\delta=1/4$; then it is not difficult to obtain that

$$D=\frac{(e+1-2e^{1/2})(1+e^{3/2}+e^{1/2})}{e+1},\qquad C=\frac{2e^{5/4}-2e-e^{1/2}+1}{2(e^{1/2}+1)},\quad \text{and}$$
$$\gamma=\frac{e^{5/4}-e^{3/4}}{e^2-1}.$$

Thus, if we choose

$$d_0=\frac{e+1}{2(e+1-2e^{1/2})(1+e^{3/2}+e^{1/2})}\quad \text{and}\quad d_1=\frac{4(e^{1/2}+1)}{2e^{5/4}-2e-e^{1/2}+1},$$

then the conditions (2.4), (2.5) and (H5′) are fulfilled. It follows from Corollary 2.1 that the boundary value problem (2.10)–(2.11) has at least three positive solutions.

References

[1] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.
[2] L. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.
[3] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.
[4] C.P. Gupta, Solvability of a three-point nonlinear boundary value problems for a second order ordinary differential equation, J. Math. Anal. Appl. 168 (1992) 540–551.
[5] C.P. Gupta, A sharp condition for the solvability of a three-point second order boundary value problem, J. Math. Anal. Appl. 205 (1997) 579–586.
[6] C.P. Gupta, S. Trofimchuk, Existence of a solution to a three-point boundary value problem and the spectral radius of a related linear operator, Nonlinear Anal. 34 (1998) 498–507.
[7] R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J. 28 (1979) 673–688.
[8] R. Ma, Existence theorems for a second order three-point boundary value problem, J. Math. Anal. Appl. 212 (1997) 430–442.

[9] R. Ma, Positive solutions of a nonlinear three-point boundary value problem, Electron. J. Differential Equations 34 (1999) 1–8.
[10] R. Ma, H. Wang, Positive solutions of nonlinear three-point boundary value problems, J. Math. Anal. Appl. 279 (2003) 216–227.
[11] J.P. Sun, H.F. Huo, W.T. Li, Positive solutions to singular nonlinear three point boundary value problems, Dynamic Systems and Applications, in press.
[12] H. Wang, On the existence of positive solution for semilinear elliptic equations in the annulus, J. Differential Equations 109 (1994) 1–7.